Abstract

The Goodwin growth model is a particular dynamical system exhibiting limit cycle behaviour. I wish to add a measure of search and selection into the basic model by adapting one of the parameters of the model to be affected by an operator, such that the search process itself is a function of the relative slackness of the labour market summarised by the Phillips curve relationship modelled within the Goodwin model, and a new operator defined below, following the Kauffman (1993) NK model of search and selection along fitness landscapes. The results of simulations show simply that the dynamics of the augmented Goodwin economy are essentially unchanged, though the search process itself is determined more by the macro-dynamics than the microeconomic conditions imposed on the actors in the system by the ruggedness of the landscape.

In physics the truth is rarely perfectly clear, and that is certainly universally the case in human affairs. Hence, what is not surrounded by uncertainty cannot be the truth.

—Richard P. Feynman

1 The Goodwin Growth Model

Goodwin (1967) developed his model of economic growth after a conversation with Sewall Wright, the evolutionary biologist, who suggested the Lotka-Volterra model of predator-prey interaction from mathematical biology would be apt as both a metaphor and method of analysis for the dynamics of a capitalist macroeconomy. Goodwin took the suggestion and modelled capitalist-worker dynamics in the following way.

Assume two homogeneous, non-specific factors of production, labour, \( L \) and capital, \( K \), where all quantities are real and net, and all wages are consumed with all profits being reinvested into the system. There is a steady growth rate \( \beta \) of the labour force \( N \) according to \( N = N_0 e^{\beta t} \), and steady technical progress, \( \alpha \) so that the capital-labour ratio evolves according to \( Y/L = \alpha = \alpha_0 e^{\alpha t} \). The capital-output ratio \( k = Y/L \) is assumed constant and the real wage rises in the neighbourhood of full employment. The workers accrue to themselves a portion of the output of the economy, \( u \) and the capitalists receive \( v \) for their efforts.

From these assumptions, and following Gandolfo (2003, pgs.449–467), we can derive the familiar and famous Goodwin equations, where \( \gamma \) and \( \rho \) are scalars:

\[
\dot{v} = \left\{ \left[ \frac{1}{k} - (\alpha + \beta) \right] \right\} v
\]

(1)
\[ \dot{u} = -[(\alpha + \gamma) + \rho v]u \]  

Goodwin, quoted in Gandolfo (2003, pgs.57-58), best describes the dynamics of this system of equations and their economic meaning:

When profit is greatest, \( u = u \), employment is average, . . . , and the high growth rate pushes employment to its maximum \( v_2 \), which squeezes the profit rate to its average value. . . the deceleration in the growth employment (relative) to its average value again, where profit and growth are again at their nadir \( v_2 \). This low growth rate leads to a fall in output and employment to well below full employment, thus restoring profitability to its average value because productivity is now rising faster than wage rates . . . . The improved profitability carries the seed of its own destruction by engendering a too vigorous expansion of output and employment, thus destroying the reserve army of labour and strengthening labour’s bargaining power.

What does this expansion and contraction of the economy look like?

![Worker's/Capitalist's Share](image)

Figure 1: Evolution of capitalist/Worker interactions as they share the products of the economy. We see here that the motion is cyclical and bounded, implying the dynamics of the system exhibit limit cycle behaviour.

The cyclical behaviour of flow of the productive resources from the capitalists to the workers and back again can be seen in a different plot, figure 1 below.

The Marxist undertones of this model have been highlighted \textit{ad nauseam}, however, I believe the reason for this is that the mainstay of the Goodwin model is the Lotka-Voltera model of predator prey interactions. The conception of the capitalist as predator sits well in the budding Marxist’s craw. Mathematically, the capitalists in this system may also be considered as the prey to a labour force with very strong hold over the means of production. The model was originally constructed to study interacting populations of fish in a mathematically rigorous way, rather than to pass some kind of judgement on the allocation of the single social product. True, Goodwin was a card carrying Communist, and it was from this standpoint that he wrote, but I feel that the model survives its
political biases, because of its generality, its novelty, and its applicability to economic phenomena in which endogenous cyclical behaviour can be generated without recourse to stochastic variables or appeals to a higher power, inside of a dynamical system that one has a hope of controlling, because the upper and lower boundaries traced out by the motion of the eigenvalues all lie in the positive orthant, generating stable, limit cycle behaviour, as we can see in figure 1.

The biological underpinning of this model is obvious. In the terminology of mathematical biologists, capital and labour are *epistatic*, that is, there exists a dependent relationship between them.

The subject of this paper is the augmentation of the basic Goodwin model with the search and selection dynamics of Kauffman’s NK model, to which we now turn.

2 Search and Selection: The Augmented NK Model

2.1 What is a Landscape?

There are two main interpretations of landscapes in biology. In the first each member of a population is considered a *point* on a fitness landscape. This imagery has been used extensively in the development of evolutionary thought . . . In the second approach a point on the landscape is taken to represent the *average fitness* of an entire population which moves on a surface of mean fitness. (Jones 1995, pg.s 5–6).

**Definition 1** (*Landscapes*) A Landscape $\Lambda$ is defined by three components $(X, \Gamma, f)$, where

1. $X$ is a set of configuration states;
2. $\Gamma$ is some notion of neighbourhood, nearness, distance or accessibility, defined on $X$.
3. $f$, a fitness function that maps $f : X \Rightarrow \mathbb{R}$.

Figure 2: Explicit limit cycle behaviour for several economies at different levels of output for the economy. Higher output implies a greater diameter of the orbit.
Given these elements and a string selection of size $N$ dependent on $K$ of its neighbouring bits, we can define a landscape with an arbitrary number of local optima, and a select number of global optima. The result is a search across a graph, but that need not concern us. The main point I want to highlight here is that there is a mechanism for describing the way agents search, and there is a means of measuring how they do it. From this, properties of the system can be adduced and once set up on a computer, the landscape can be simulated, calibrated, and predictions can be drawn.

The NK model was developed by Kauffman (?) and others to explain genomic selection in terms of a search for an increased level of fitness, where a level of fitness is a local optimum. The theory allows one to change the ruggedness—the number of local optima—and thus tune the landscape to accept any arbitrary level of complicatedness. I use complicatedness, rather than complexity, because ‘complexity’ means something very specific in this model.

The number of local optima is positively related to the complexity ($K$) of a system. Even for small values of $K$, finding the global optimum requires exhaustive search also called global search $2^N$. Kauffman (1993) observed that the fitness of local optima tends towards the mean for large $N$ and $K$ values. Kauffman refers to this as a complexity catastrophe.

### 2.2 Previous NK Studies in an Economic Context

Several theoretical and empirical conclusions have related the NK model to bounded rationality (Frenken, Marengo & Valente 1999). They find that in evolutionary worlds survival depends on short run profits. Local search (leading to local optima) performs better than global search (required to find the global optimum). Their work supports the assumption of local search. Rivkin (2000) focused on imitation of complex strategies, where the more complex a technology or business, the more firms need to rely on innovation rather than on imitation. His work supports the evolutionary.resource-based/competence view on non-mainstream economics. The NK model has also been used to model changes in technological paradigms (Frenken 2004). As the dimensionality of technology grows over time, early developed components become rigid (e.g., the gasoline engine). Thus the NK model has been used to develop an alternative notion of lock-in.

Some empirical results have been made in the area of recombinant search (Fleming and Sorenson 2001). $K$ is made operational by a measure of how often two patent classes have been combined previously. The more often, the lower the complexity. They then test whether a large $K$ means large dispersion in success rate (as measured by citation rate). Radical innovation can now be understood as innovation by recombining previously unconnected technologies. The NK methodology has also been applied to steam engine designs (Frenken 2004). They reconstruct the design space, and then analyse whether the competing technologies substitute or co-exist (reflecting local optima). This finding demythologises the simple linear succession history arguments of neoclassical economic historians like O’Rourke & Williamson (1999).

### 2.3 Pseudo-code Implementation

1. **Initialization** Generate an initial set $S = \{s_1, \ldots, s_m\}$ of $m$ random character sequences, each of length $N$. Each $s_i = s_{i1} \ldots s_{iN}$, $i \in \{1, \ldots, m\}$, takes its elements from the alphabet $\alpha = \{A, B, \ldots, Z, \}$.

2. **Evaluation** For each individual $s_i$ compute its fitness value $\tau(s_i)$, which is its *Hamming distance* to the objective sentence $s^{\text{obj}}$.
\[ \tau(s_i) := \sum_{k=1}^{N} (\epsilon(s_{ik}) - \epsilon(s_{k}^{\text{obj}}))^2 \] (3)

Here \( \epsilon \) refers to the encoding function which returns an integer from the set \( \{65, \ldots, 93\} \) for each element of alphabet \( \alpha \).

3. Selection and mutation Select the best evaluated sentence \( s^* \) with \( \tau(s^*) = \min\{\tau(s) | s \in S\} \), i.e., the one closest to the objective sentence, and generate \( m-1 \) mutated offspring from \( s^* \).

4. Termination check If the objective sentence has not yet emerged, that is, as long as \( s^* \neq s^{\text{obj}} \), continue with step 2.

2.4 An NK Example

As an example of the NK model, let us generate an arbitrary string, and one whose optimum is known. From the Roman alphabet, our \( N \), we pick some ordered sentence, say, GARYMONGIOVI which here represents the optimum that we wish to attain: it is the most highly organised state, by our lights. GARY MONGIOVI is thus the optimum we wish to reach, and the alphabet is our set of configuration states, \( X \). It is fairly obvious that \( X \) is a large number, even for a 12 bit string like GARYMONGIOVI, the set of all possible configurations is of the order of \( (26!)^{12} \), or \( 1.8510933542 \times 10^{319} \). For each character there are 25 possible neighbours, and so \( K = 25 \).

We take our string and change one of the characters in the string one at a time (this is how we move inside the landscape) until the string converges to the optimum. The notion of ‘nearness’ here is the Hamming distance operator, which is not the usual measure. In ‘standard’ landscape theory, the distance or neighbourhood measure is the average of the component functions,

\[ f(x) = \frac{1}{N} \sum_{i=1}^{N} f_i(x_{i1}, x_{i2}, \ldots, x_{iN}). \] (4)

We use the Hamming distance, defined above, rather than the standard measure, because it allows a more economic interpretation of the cost function. Our firms will be searching a configuration space of production possibilities, changing one part of their existing production possibilities incrementally until a new level of fitness is reached. The least squared distance between one production level and another leaves one with a more natural interpretation of the cost of change. It also mitigates against a firm starting with very low organisational capital simply ‘leaping’ to the status of an IBM, say. The implementation is basically a hill-climbing algorithm on a fixed state space. Thus with a fast computer and a decent implementation, it should converge to the optimum quite quickly.

When we run the model, we can see in figure that the search converges very quickly. It takes about 50 generations to reach the optimum sentence.

So: the plan is to splice the two models together, to give the Goodwin model some more interesting dynamics as a result of the search process over time.

3 Search and Selection, an augmented Goodwin Model

The sequence of events in this economy is this: there is a list of profitable opportunities to be searched through. The opportunities are generated randomly and the agents have no prior beliefs about the types of opportunities they might find, which changes the capital labour ratio overall as
Figure 3: Here we see the square of difference from the actual values to the optimum goes to zero after about 50 generations. Not bad, given the size of the state space.

the pool of available opportunities decreases—there is a limit to profitable availability of these opportunities. First, employers and workers start with a randomly allocated amount of organisational capital. Second, capitalists must now decide to change their productive process or keep it the same. Each capitalist adopt a new productive process with certainty. The cost to the capitalist is the Hamming distance from their original position in the technology landscape. If the cost of the search is too great for the capitalist to endure, they go out of business, or die. There is no credit market in this model, as the Goodwin model assumes all profits and savings are reinvested, so at the end of each time period, the firms that survive so so relative to a local optimum. Because work is assumed to produce disutility, labour power is inversely proportional to worker efficiency. The workers want the most of the share of the resource they can get, while working as little as possible for it. The capitalists have the same objective, except they wish to maximise worker efficiency as well as their share of the resource. Each productive process has a level of worker efficiency associated with it. The capitalist must choose each of these levels inside the prospective productive resource plan, and then decide whether to move to that level of output for that level of efficiency. Once the choice is made, the firm moves to that point of the production plan landscape. If the firm is not at the local optimum for the system, the process repeats.

Intuitively, it makes sense for the firm to search far away on the technology landscape when labour efficiency, $\alpha$ is low in order to sample beyond the correlation length (here the Hamming distance) of the configuration space. As labour efficiency increases, optimal searches are confined to nearer to home. So we expect to see slingshot dynamics, where the search lengths taper back quickly, shrinking the output of the economy rapidly around one point, which is what we see in the numerical analysis below.

Formally, let us define our landscape. We have a measure of the composite good to be allocated between each of the two classes, $X$. There are $N$ distinct production plans, $\kappa$, which takes elements from the set of total available production plans in the search space, $\Pi$ and maps it onto the real line in the (strictly) positive orthant:
There are discrete choices of operations inside each production plan, $\kappa_i^j = \{1, \ldots, S\}$, and it is these that will be changed as the firm moves about the landscape. Each production function is composed of these operations:

$$\kappa_i = \{\kappa_i^1, \ldots, \kappa_i^j, \ldots, \kappa_i^N\}$$  \hspace{1cm} (6)

It is obvious that the absolute value of the total number of configurations, $S$, has to equal the size of the search space, so $|\Pi| = S^N$. Now an important point: the level of labour efficiency is tied at each point on the technology space to an operation, and once the firm discovers this point in the space, the level of labour efficiency for each configuration, $\phi_j^i$ is known with certainty and maps according to

$$\phi_j^i = \phi_j^i(\kappa_j^i, \kappa_i^{-j}),$$  \hspace{1cm} (7)

which says that $\phi_j^i(\kappa_j^i, \kappa_i^{-j})$ is the payoff to the jth operating unit when it is state $\kappa_j^i$ and the other operations are in the state encoded by the vector $\kappa_i^{-j}$. Each configuration is conserved, that is, a configuration is fixed and fully known once discovered, and implemented with certainty:

$$X(\kappa_i) = \frac{1}{N} \sum_{j=1}^{N} \phi_j^i = \frac{1}{N} \sum_{j=1}^{N} (\kappa_j^i, \kappa_i^{-j}).$$  \hspace{1cm} (8)

Nearly there now. Each of these operations, $\kappa_i$, are connected. One cannot change half of a productive process without effect, and a measure of the ‘interconnectedness’ of each of the productive elements is necessary. Define $K_j^i = K(\kappa_j^i, \kappa_i^{-j})$ as this measure of ‘interconnectedness’ or neighbourhood effect. The value this takes is

$$K_j^i = \begin{cases} 1 & \text{if setting of operation } j \text{ affects the labour requirement of operation } k \\ 0 & \text{otherwise} \end{cases}$$  \hspace{1cm} (9)

The state variables for the system are $k$, capital per capita and labour efficiency, $\alpha$. Everything else in the system is both parametrised and fixed, as in the Goodwin model specified in equations 1 and 2. I add a third equation to this system of coupled equations to take account of the search process and to explicitly measure the amount of capital and labour being used up or ‘wasted’ in the search process. One can compute a welfare loss to this economy by simply measuring the output levels of the economy at different levels of ruggedness of the search landscape and comparing this to the unaugmented levels of output of the 2-equation Goodwin model (in polar coordinates). Once the search process terminates, a value of $k$ and $\alpha$ is given to the system through equation 10 for each generation, and the system’s dynamics are altered correspondingly.

The third equation in the system is

$$\dot{k} = (-ek + fv - \alpha)uk$$  \hspace{1cm} (10)

and, added to the familiar Goodwin model equations, makes the system under study. Many simulations were run at different levels of $N$ and $\kappa$, and the dynamics of the system were largely similar to the Goodwin model in most respects. The basic stability properties of the Goodwin model are unchanged. We are still in the positive orthant, so there will be a limit cycle after some time away from the equilibrium of the system. The fluctuations in the system are coming from
the search process, as we can see. Figure 3 was generated with the following parameter values: \( r = 1.1, d = 0.2, \epsilon = 0.3, f = 0.2, p = 20; x_0 = 1, z_0 = 1, y_0 = 1. 

Figure 4: There are 3 lines here. The top line is the capitalist’s share of output. The middle line is the hamming distance as a function of capitalist’s share of output. The bottom line is labour’s share of output.

And we can see from the phase diagram that the system is stable. This turns out to be true for quite a wide range of parameter values.

Figure 5: Stability of the new three equation dynamical system.

What have we learned about the dynamics of the system? Not much, as it turns out, but the importing of a search and selection process into the Goodwin model is a good place to start, as it carries with it explicit and implicit assumptions about how the firms and labourers behave in the system. The model generates cycles endogenously as it did before, and now that the firms have to search for new capital projects, overall welfare is lessened, as one can see by subtracting the interpolated values of the Goodwin \( x \) and \( y \) equations from the new \( x \) and \( y \) equations. We get a positive number, and this is the welfare loss associated at each point with the inclusion of the search process in terms of the representative good.
4 Goodwin, Selection, and Post-Keynesian Economics. Discussion and Further Work

Post-Keynesian economists believe that a capitalist economy has no natural or automatic tendency towards full employment. This model incorporates a notion of full employment, but this equilibrium is unstable—the next period will see the economy shift from full employment as a consequence of the dynamics of the system. Post-Keynesian economists believe fixed investment is a major determinant of the level of aggregate demand in closed or large economy. The augmented Goodwin model obviously incorporates a measure of investment, as all profits and wages are reinvested and the proceeds go towards the acquisition of the next period’s productive processes. The level of investment is fixed at 100%, an unrealistic assumption, but to allow savings and investment into the system would be a complication best left to another paper. Post-Keynesian economists believe decisions on the level and direction of investment are made in anticipation of future events, which agents cannot know even probabilistically. They emphasize the role of uncertainty as being central to economic life, an observation shared by physicist Richard Feynman, quoted above. The augmented Goodwin model is deterministic, generating limit cycle behaviour for a large range of values, before the dynamics explode out beyond the neighborhood of the system. Uncertainty is present in the search process, where a firm may search too long, incur higher costs, and fail to produce enough to survive. It has no knowledge of the ruggedness of the landscape, and no other information to guide its search than the past path it has taken, and its cost function. In that way uncertainty, rather than risk, is present in the system at its core. The Post-Keynesians emphasise the need for government fiscal policy to support institutions to support employment and incomes. There is no government in this model, though a reduction of overall output and therefore overall welfare is present, a government function which blindly introduces fiscal spending when heading into a recession and reduces spending heading out of the recession could be included in the augmented Goodwin model easily. Time and its effect on economic problems is also emphasised in Post-Keynesian literature. This model has explicit dynamics—the model is implemented as a set of coupled difference equations which evolve over time. The equilibrium position of the system is determined by the parameter values chosen, but limit cycles are present, so the system will definitely cycle around an equilibrium set of points in phase space. Thus this model has a Post-Keynesian hue, and predicts largely what Post-Keynesian economics would predict—increasing uncertainty by making the search landscape more difficult to traverse will increase the likelihood of economic collapse. Policies that reduce uncertainty will increase overall welfare, while policies that decrease labour bargaining power will not. Labour bargaining power is a function of each productive plan being used. Because this is a linear relationship, the lower bargaining power is, the lower the level of productivity will be relative to the level of productive technology, and so one forces the other down.

4.1 Further Work

There are several ways to extend the model:

1. Economic parameters. The Goodwin model is not able to take account of standard economic parameters (Choi July 1995). Introducing real world data and re-estimating the model will give it more relevance.

2. Savings and investment. Introducing saving and investment or lagged investment effects will almost certainly destabilise the model, but the model will be more realistic.
3. Labour bargaining power. Measuring the level of labour power consistent with pre-determined levels of output would be a very interesting place to start.

References


