The Methodology of a Computable Economist*

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Abstract

Computable economics is the formalisation of economic theory in effective ways (Velupillai, 2000, pp. 10–11). For a coherent, computationally tractable model of human behaviour to be effective, a finite set of steps leading to its solution in finite time must be explicitly given or be at least theoretically possible. A computable economist interprets economic data digitally, and proceeds through experimentation, simulation, and estimation towards careful policy prescriptions. What does this imply for the methodology of a computable economist? How should she go about her professional duties by doing economics in a computable mode, and how is that different from more mainstream approaches?

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One of the great insights of twentieth-century logic was that, in order to understand how formulae can bear the meanings they bear, we must first strip them of all those meanings so we can see the symbols as themselves. Stripping symbols of all the meanings we have so lovingly bestowed on them over the centuries in various unsystematic ways seems an extremely perverse thing to do – after all, it was only so that they could bear meaning that we invented the symbols in the first place. But we have to do it so that we can think about formulae as (perhaps mathematical) objects in their own right, for then can we start to think about how it is possible to ascribe meanings to them in a systematic way that takes account of their internal structure. Thomas Forester, *Logic, Induction, and Sets*

## 1 What is Computable Economics?

If modern economics is the study of how we as a collection of individuals choose to allocate a finite amount of resources among competing expected future states of the world, then *computable* economics asks how many of those allocations that we as a society of individuals might choose have an explicit set of steps leading to a solution, even in theory. In other words, how many of these future states of the economic system are ever possibly realisable. The choice to do modern economics computably is founded on a branch of mathematics—recursion theory—developed in the last century to grapple with the question of what could be achieved by computation with finite means. Recursion theory begins by defining sets around zero, one, and operations like plus and minus, thus locking the choices from such sets into the realm of computable numbers. Then a tool for computing the choice is used: the Turing machine, or a variant of this. For detailed descriptions of Turing machines and their relationship to economic theory, see Velupillai (2000, pp. 185–197). The computable economic analysis I describe and allude to here is explicitly constructive, that is, the models are formalised using the tools of recursion theory to ensure that an algorithm can be found to implement the model in a consistent way on a computer. The main advantage of doing economics in this way is to improve the accuracy of calculations made using a computer, as all modern economic computations are. In contrast with mainstream mathematical economics, which relies for its results on the sometimes non-constructive (i.e., uncomputable) mathematics of set theory, computable economics, by definition, is a set of techniques designed to give more realistic answers to policy questions. Many of the standard theorems of mainstream mathematical economics use undecidable disjunctions like the axiom of choice in the formulation of their proofs. The inclusion of the axiom of choice creates several problems for the economist interested in applying these results to the study of the real world. To the practical, policy-oriented economist, results obtained on a computer using models with uncomputable theoretical foundations must be circumspect. To the academic interested in developing models of the real world, the same criticism of inaccurate calculation is apposite. Using proof techniques like the axiom of choice ensures that no algorithm can be created to apply the theorem over an arbitrary domain and range. The use of non-constructive methods in the proofs of these theorems makes it impossible even in theory

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1. To name but a few: The Bolzano-Weierstrass theorem, McKenzie’s theorem, and Brouwer’s Fix-Point theorem.
2. Often stated as: let $C$ be a collection of nonempty sets. Then we can choose a member from each set in that collection. In other words, there exists a function $f$ defined on $C$ with the property that, for each set $S$ in the collection, $f(S)$ is a member of $S$.
to design a series of steps leading to the solution of, say, the standard Arrow-Debreu general equilibrium problem, because although it can and has been shown that an equilibrium exists, Arrow and Debreu (1954), there is no way to construct a trading path to this equilibrium position for arbitrary numbers of traders for arbitrary initial trading positions in a finite number of steps. In this sense, modern economic theory is uncomputable in large measure. What then does it mean to compute an economic ‘object’, where object in runtime, such as the equilibrium in a market?

1.1 To ensure an economic object can be computed, model it recursion-theoretically

A set of numbers, $S$, is called recursive if the question “is some number $x \in S$?” can be answered in a finite number of steps, that is, the question is decidable. For Gold, ?, pg. 29, the concept of a limiting decision procedure is key to the use of recursion theory in attempting to answer questions posed inductively. Here a guessing function, $g(x, n)$, is specified to compute successively larger amounts of information up to infinity, allowing this guessing function ever-closer approximations to the correct answer.

A computation is a process whereby we proceed from initially given objects, called inputs, according to a fixed set of rules, called a program, procedure, or algorithm, through a series of steps and arrive at the end of these steps with a final result, called the output. The algorithm, as a set of rules proceeding from inputs to outputs, must be precise and definite, with each successive step clearly determined.

(Soare, 1999, pp.6) When we compute an economic macrostate, like the equilibrium value of supply and demand schedules, we are asserting that, given a small perturbation of the initial conditions given to the system, something close to the original equilibrium will be attained again (this assertion is of course not always definite, chaotic dynamics). The specification of an economic model recursion-theoretically means that one models economic agents as Turing Machines, extremely powerful theoretical devices capable of computation and the simulation of rational behaviour (Velupillai, 2000, pp. 37–40 and pg. 51). At first blush, the problem of Why do the tools that modern economic analysis is built upon have gaps in them that prevent their usefulness? One explanation Velupillai (2000, pg. 12) is that it simply happened by historical accident:

There are no compelling doctrine-historical, analytical, or descriptive reasons for preferring a formalism of set theory and model theory rather than, say, recursion theory. It is my contention that is was entirely an accident of the history of ideas that the mathematisation of economics took the path that led to a basis in set theory and model theory.

Although I am inclined to agree with the above quote, whether or not a distinct set of mathematical tools was deliberately chosen or not, with the demonstration of the incompleteness and fundamental uncertainty inherent in any system of mathematical logic by Gödel, Turing, Kleene, Church, Markov and others, the set of tools used by standard economic theory has to be brought into line with these developments, because fundamentally, the development of theoretical tools to understand uncertainty, incomplete information, irrationality and finiteness concern those concerned with the analysis of economic life.
1.2 An example

Consider the following: we are given to suppose that the evolution of a specific economic system, say the price vector of a set of homogeneous commodities, in the traditional undergraduate-level supply and demand model, is given by a strictly continuous differential equation of the form

$$\dot{x} = (\pi, x; A, t),$$

where $x$ is a vector of prices, $\pi$ a vector of prices of goods for sale, and $A$ a transition coefficient matrix with each element time-denominated. We are asked whether it is appropriate to use this ordinary differential equation (and an appropriate functional form) to model the movement of prices towards or away from some equilibrium set in our specific example. Given that the economic data we will use to test this hypothesis is discrete, we will want to discretise the flow into a map, that is, we will want to analyse the motion of the system as an equation of the following form:

$$x_t = x_{t-1} + \pi x_{t-1} + Ax_{t-1} + C.$$  (2)

Here $C$ takes the usual interpretation. It is natural to want to trace out the implications of such a discretisation, both in theory and in practice. Why are we doing this? Because, although most of the theoretical results on existence and stability of equilibria are defined for differential equations, difference equations readily yield to approximate numerical solution via various methods described below. Solutions to difference equations are obtainable by sequential arithmetic operations, a task for which the computer is precisely suited. The approach of numerical analysis is to find an efficient algorithm to approximate solutions to the difference equation, which may not have an explicit analytical solution in any case, particularly if the difference equation takes a non-linear form.

The next step is to ask how accurate such a numerical approximation is. Taking equation 2 and some initial seed point $x_0$, compute the Taylor series expansion of $x_t$ around $x_0$:

$$x_n^T = (C + Ax_{t-1} + \pi x_{t-1}) + (Ax'_{t-1} + \pi x'_{t-1}) t + \left(\frac{1}{2} Ax''_{t-1} + \frac{1}{2} \pi x''_{t-1}\right) t^2 + \left(\frac{1}{6} Ax'''_{t-1} + \frac{1}{6} \pi x'''_{t-1}\right) t^3 + O_t^4$$  (3)

where $O_t \in [x_t, x_T]$ is the remainder term. Equation 3 is substituted in for $x_t$ to find the approximate solution to $x_t$. A natural question to ask at this stage is whether the numerical method picks out an approximate solution that converges on the exact solution over the interval $[x_t, x_T]$. An approximate method is said to be $T$ order exact if the remainder term disappears when the over the finite interval $[x_t, x_T]$. This approximation is then said to be convergent Stuart and Humphries (1998). The idea is that through successive approximations, in the limit, a description of the true behaviour of the system is gathered. In every one of these approximation methods, however, there is a local truncation error of size $E_{n+k}$ with weights on the jth iteration $\alpha_j$ where, at the nth step of the series expansion,

\[\text{See } (? \text{, pgs. 258–206) for a treatment of excess demand functions and their applicability to studies of decision problems in general.}\]
\[ E_{n+k} = \sum_{j=0}^{k} \alpha_j x(t_{n+j}) - O \phi(x(t_{n+k}), x(t_{n+k-1}), \ldots, x(t_n, t_n; O)). \] (4)

and the local unit truncation error is given by

\[ \theta_{n+k} = \frac{E_{n+k}}{O}. \] (5)

These errors are local errors: errors that accrue in the computation step by step. The total error in general explodes as one strays further from the seed value where the computation began. There are two components to this error: the finite numerical accuracy of the calculation, which is dependent on the data of the problem, and the computer software’s method of rounding off decimal places, often called rounding error. The total truncation error in the system is the difference between the numerical solution of (2) for prices and quantities, \( x(t_n) \), and the actual solution calculated after \( n + 1 \) steps, \( x(n + 1) \).

So, in the estimation of our simple model, we have two main sources of error—the data themselves gathered from the real world, and the method the computer must use of approximating the difference equation which we have derived from theory. This is the problem faced by every quantitative economist in the profession, whatever their specific research question.

How would a computable economist represent the supply and demand problem so as to reduce the prevalence of these errors?

2 Economic data is fundamentally discrete

The main feature of economic data like quarterly GDP estimates and so forth, is their discreteness. The sale of a large, durable good like a house is at best an annual or semi-annual event. The use of time series to measure the effects of one system upon another has their use predicated on the ability of high-frequency data with a high signal to noise ratio to explain variation in one another. This does not work when data is not at a high frequency and is plagued by measurement error and aggregation biases. But every reader knows this. What is so different about viewing data in a manner that makes them amenable to analysis using computable economics? For an orthodox economist, economic problems are primarily defined as quantitative questions, such as the incidence of taxation, or the effects of a devaluation, depending for their ‘solution’ on the collation of differing pieces of information, the fitting of a model to some data collected by an outside agency like the State.

John von Neumann et al. (1946, p. 1022) gave us a look at what types of questions economic data can ask, given the wide margins of error one introduces when one is really looking at the data and working from it. I have in mind quarterly or annual national accounting data. As he sees it, data is characterised by five main sources of error:

1. data is discretely observed and prone to large errors in measurement, these errors come from many sources which we listed above;
2. the parameters which fit idealised models to data are subject to the same measurement errors;
3. the use of numerical methods in approximating the model adds more error to the system;

4. the use of a computer (human or machine) tasked with actually computing the answer to the stated model will introduce errors through the reduction of implicit formulations of equations and transcendental operations to ‘simple’ operations the computer can actually perform;

5. round off errors in digital systems (and noise in analogue computing machines) will cause still further error.

von Neumann’s criticisms seemed to have fallen on deaf ears. Griliches Griliches (1986), writing nearly 40 years later in The Handbook of Econometrics, regards the essentially the same set of problems and the set of ad-hoc solutions he proposes to deal with them as the true work of the applied economist or econometrician. The profession has not gotten very far in its endeavours to tame the errors inherent in economic data it seems, despite many economists’ efforts to bring computational tools and numerical analysis to the fore in economic reasoning.

Let me state von Neumann’s errors in terms of the supply and demand problem posed above. We started off with an analytical model of market-clearing price determination, given in equation 1, then moved to a discretised version of it in equation 2. We computed a Taylor series approximation on the difference equation, and (presumably) solved an approximate version of that equation numerically. So, we have gone from an idealised model of the underlying problem, expressed with idealisations and simplifications, to the parameterised difference equation version, where the measurement error of the parameters $A, \pi$ and $x$ will play a part in determining the solution value of the price vector $x$, to an estimation using the computer, where transcendental functions and numbers (like sin, cos, log, etc, depending on the functional form used) will be approximated using numerical methods, and the computer will arrive at an approximate solution using some particular convergent method.

It is obvious that in tracing out the path from stating the model to the discovery of a numerical ‘solution’, we replaced the strict mathematical equivalence of equation 1 with an approximate one by the time we come to equation 3, and an even more approximate answer is the result of the numerical estimation of this equation in some particular functional form. I would like to focus briefly on the role of truncation errors in economic data as a preface to the solution to these errors with a different kind of modeling.

### 2.1 Truncation errors are the result of both mis-specified models and computational limitations

The data generated by economic life will not change unless economic life changes, though the measurement of these phenomena might become more precise. What may become more apparent is the increasing use of ad-hoc reasoning in modeling economic systems and the truncation errors that plague them, giving false roots to time series estimations and throwing off predictions and forecasts. When a continuous differential model is discretised, the discretised version of that model need not be stable any longer, and in fact, it is only for a restricted class of systems that such stability is guaranteed, because the necessary and sufficient condition for the stability of the finite differenced approximant of equation 1, equation 3, requires for stability that

\[
\Delta x \geq \left( \frac{dx_t}{dx^*_t} \right)^{\frac{3}{2}} \Delta t.
\]
Equation 6 says that the changes we subject the price system to ($\Delta x$) must be relatively small relative to the time increments we are looking at, and we must have data on the entire domain of integration before we can be assured of stability. This is very difficult to assure in practice. But, being aware of the problem, a computable economist, if she wants to use this methodology, must have an argument that extends the stability properties we know exist in equation 1 to the approximation of that problem, equation 3, or truncation errors will ensure that the slightest perturbation of the initial data will remove any predictive power the model might have seemed to have before the perturbation was attempted. Most economists do not go through this stage, however.

The source of a truncation error can a non-repeating decimalisation when the number, say, $1.333333\ldots$, is converted to its binary analogue, $1.0101010101010101\ldots$ to be processed by our computer, or it can simply be a rule of thumb designed to keep computations within acceptable bounds of efficiency. The simplest method of truncation is simply to omit all digits beyond some point $s$ in the binary description of the number in the computer. The computer performs a maximum precision calculation on this number—usually back to 256 places, it depends on one’s software—and the computation continues. Computers are not designed to deal effectively with irrational numbers, and don’t do well when placed in a situation of having to evaluate the effectiveness of irrational number-laden data with respect to the undecidable elements in the models such data are being fitted to. The individual operations of the computer while doing this fit are subject to truncation errors, which are cumulative over the computational cycle. When such operations are repeated in large numbers of computations, as in CGE modelling, the answers such models pump out and the policy conclusions drawn from them need to be foreshown. John Rust (1998), castigates Computable General Equilibrium (CGE) models for these shortcomings:

> The reason why large scale computable general equilibrium problems are difficult for economists to solve is that they are using the wrong hardware and software.

To add fuel to the fire and, in a sense incinerate the CGE modeling industry, Velupillai has shown formally that the CGE model is neither constructive nor computable in the formal sense of either.

In tracing out the path from stating the model to the discovery of a numerical ‘solution’, we replaced the strict mathematical equivalence of equation 1 with an approximate one by the time we come to equation 3, moving in ever-increasing circles of technicality. The first source of error, the model itself, is rather easy to remove—just use another model, if one is available. The second source of error, the measurement of data and the parameters that result, depend on the details of the data collection. When we have to look at the approximated differenced version of the model, stability concerns us as well as the mathematical approximations used. When we get down to truncation errors, we are really talking about the algorithms the computer is using to perform the calculation. A lot is being demanded of our computable economist, if she is to take account of all of these issues and still say something meaningful in her modeling exercise.

### 2.2 How stable is the answer the model gives?

Let us say our computable economist is aware of these shortcomings of modern models. How is she to react? Firstly, an interpretation of the sources of error is important to have. It should be
stressed that no specific theory, neoclassical or otherwise, is being taken here—yet. The errors due to theory are another day’s work. Once made is aware of the scale of the errors due to observation, the computable economist must ask the question: what are the limits of the change of the result, caused by the change of parameters estimated with the data of the problem? How stable is the solution the computer has just output? If the source data is perturbed by a small amount, will the answer change radically? If so, a policy based on such an answer is likely to be equally unstable. The stability of the problem must be confronted first when trying to reconcile models and their predictions to error-prone data.

An example from development economics may help at this point. It is well known that cross-sectional growth regressions have serious limitations in terms of their estimations of the effectiveness of aid on economic growth, controlling for several explanatory variables like autonomous investment, educational levels, and so forth. Apart from the usual econometric concerns about endogeneity, outliers, model uncertainty, and measurement error, a key drawback is the problem of unobservable heterogeneity or the omitted variables problem. In cross-country regressions, we can never be sure whether we are controlling for all possible ways in which countries might differ. A dominant concern in the recent literature has been the lack of robustness of estimates of the effectiveness of aid on growth. Levine and Renelt (1992) showed that growth regressions are generally quite non-robust to variations in the set of conditioning variables by a simple perturbation analysis. They took the standard model and estimated it. Then they perturbed each data point slightly, and re-estimated the model, only to find some variables became insignificant while others changed signs. Finally, they added new variables systematically to the system, and saw the growth regressions lose their robustness. Their conclusion was for a more restricted and careful usage of these econometric tools when applied to the debate on aid and growth. This criticism of the data has spawned more than 150 papers testing for the stability of these regressions with varying degrees of success Hansen and Tarp (2001).

So where is our computable economist to go? She is aware of the difficulties of estimating models without appropriate computability constraints and the inherent instability of many of the solutions to her regressions. I argue in the next section that the way forward from this apparent impasse is recursion-theoretic modeling of economic agents coupled with agent-based simulation and estimation via fundamentally inductive methods like Rissanen’s Minimum Description Length Principle. Really what all this implies is a return to induction as the basis for the analysis of economic phenomena and taking Babbage’s calculating machines seriously.

### 2.3 Problems Must Dictate Methods

**Turing** You have a logical system, a system of calculations, which you use in order to build bridges. You give this system to your clerks and they build a bridge with it and the bridge falls down. You then find a contradiction in the system... You cannot be confident about applying your calculus until you know that there is no hidden contradiction in it.

**Wittgenstein** There seems to me to be an enormous mistake there. For your calculus gives certain results, and you want the bridge not to break down. I’d say that things can go wrong in only two ways: either the bridge breaks down or you have made a mistake in your calculation—for example, you multiplied wrongly. But you seem to think that there may be a third thing
wrong: the calculus is wrong.

**Turing** No. What I object to is the bridge falling down.

**Wittgenstein** But nothing need go wrong. And if something does go wrong—if the bridge breaks down—then your mistake was of the kind of using a wrong natural law.

(?, pp.216–218)

The above quotation of an exchange between A.M. Turing and Ludwig Wittgenstein encapsulates, for me, a large part of the discussion between more or less ‘standard’ economic approaches to modeling problems in the economic sphere, and the computable approach. The standard (i.e., neoclassical) approach is primarily deductive, naively positivist and Bourbakian in their attempts at ever-increasing formalism. They seek laws for which

### 3 Induction and Simulation of Macrostates

For the purposes of this paper, I define induction as the process of discovery of rules which compactly describe past observations in order to imply predictions of future observations from the application of the rules. As economists, we are interested in generating descriptions of behaviour from the data we collect or have collected for us. Traditional models built on naive positivist deductionist foundations, use set theory and related theoretical technologies to do the heavy lifting of the modeling. I argue here that the characterization of economic agents as complex entities capable of learning from their environment must proceed from an inductive starting point via simulation and estimation of recursion-theoretic models to avoid the modeling problems I highlighted in sections 2 and 3.

How can casting our example in a computable mould avoid the traditional inductive problems of infinite regress of predicates? The main methodological advantages of such a reconstruction are clear if

- the evidence for the inductive inference is characterised recursion-theoretically and,
- the mathematical rules confirming the evidence are also specified as algorithms.

This formulation bypasses the thorny methodological issues one must deal with when using induction in set-theoretic spaces, like the problem of infinite regress. Recursion theory has been called ‘concrete mathematics’ for the very reason that it explicitly avoids any appeal to infinities or non-constructive methods. Casting models in a recursion-theoretic framework also allows a more unified construction of simulation experiments in this light. A simulation is a theoretical mechanism which allows the modeler to imagine what happens when, starting from some known initial condition, interacts the states of individual subsystems through many predefined transitions. From this the simulation itself generates the observable, possibly dynamical, phenomena we see as the macrostates of the system. The representation of the system and its dynamics are inherently implicit and constructive when the model is specified in recursive terms. It is important to note that the relations that are described by the various macrostates of the system are *not* encoded in the components of the simulation, though they are ‘computed’ as the procedure generating them proceeds. It is useful to think of a macrostate as a ‘ledger’ of the system created by the motion of
the elements in the system, rather than something that can be inferred from the initial conditions. Standard models of individual behaviour model the individual as a simple function or a set of rules that are being followed to some particular end. I argue that this conception of the individual as a simple object is misleading: the individual in an economic system is the most complicated element, while the macrostates they generate as a by-product of their coordinated behaviour are relatively simple, especially in description. Whether this procedure terminates in a satisfactory way is not for me—or anyone else—to say, because of the Halting theorem. These macrostates are observed to evolve or emerge from the interactions of the individual—thus the collective effect observed is formed by but is crucially different from the individual behaviours of the system components. Along the way, the researcher must make choices about what to simulate: the simulation is a set of synthetic methods, because one uses analytical tools (maximisation, calculus, and so forth) without necessarily looking for tractable analytic solutions to the problems posed. Indeed, analytical solutions may not exist in these models at all, and will not once sufficiently complex dynamics have been observed in the system. I contend that the choice of what to simulate will follow be affected by the formalism as much as by the availability of observable data. I would like to finish this paper with a quote from Steve Smale (Smale, 1976, pg.290), the great dynamical systems theorist and topologist, who, when asked to consider the problem of general equilibrium, had this to say:

We return to the subject of equilibrium theory. The existence theory of the static approach is deeply rooted to the use of the mathematics of fixed point theory. Thus one step in the liberation from the static point of view would be to use a mathematics of a different kind. Furthermore, proofs of fixed point theorems traditionally use difficult ideas of algebraic topology, and this has obscured the economic phenomena underlying the existence of equilibria. Also the economic equilibrium problem presents itself most directly and with the most tradition not as a fixed point problem, but as an equation, supply equals demand. Mathematical economists have translated the problem of solving this equation into a fixed point problem. I think it is fair to say that for the main existence problems in the theory of economic equilibrium, one can now bypass the fixed point approach and attack the equations directly to give existence of solutions, with a simpler kind of mathematics and even mathematics with dynamic and algorithmic overtones.

References


