Lecture 10. CAPM

Stephen Kinsella

First, we have to describe the market properly. This was first done by Sharpe in 1963. The idea behind Sharpe's paper is stocks tend to move up or down with the market itself. Of course they would, being a part of the market themselves. The insight which won Sharpe the Nobel prize was to quantify this movement in one measure, which we now call beta (β). Sharpe and his co-developers were developing on an idea first put forth by James Tobin in 1958, who showed that when all assets are risky, the efficient frontier the valuation of these assets supply is the simplest explanation of the behaviour of the market.

Say there is a linear relationship between the expected value of an asset and the market as a whole. This relationship would be described by the function

$$E(R_i) = \alpha_i + \beta_i E(R_m). \tag{1}$$

The equation above says that the expected return on an asset, $E(R_i)$, is determined by some constant interval value which varies between securities, α_i, and the 'beta' coefficient times the expected return on the market. The beta, then, is a measure of how sensitive the individual stock is to changes in the market as a whole.

If $\beta > 1$, the asset is more volatile than the market. If $\beta = 1$, the asset will move with the market in lock step. If $\beta < 1$the asset will fluctuate less than the market as a whole and will have a lower rate of return.

The beta relationship is now a benchmark in finance, and we'll spend some time in class discussing it's usefulness and it's pitfalls.

Deriving the CAPM

Let's begin from the observation that, in Expected Return-Price space, for increasing levels of efficient mean-variance combinations, we will have a straight line. This line is called the Captial Market Line, or CML. The CML captures the effects of increasing mean-variance combinations on the efficient (that is, market) portfolios.

The CML is the straight line in the figure below.

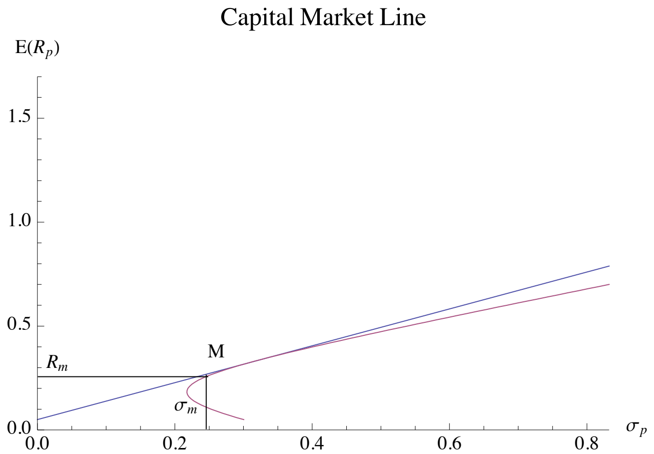

Figure X. The Capital Market Line.

For any point on the CML, there is a portfolio of stocks and shares held by individual investors which is dominant and tangential to the CML line. This portfolio is the market portfolio. At the point of tangency between the CML and the market portfolio, the slope of the CML is given by

$$E(R_i) - E(R_f) / \sigma_m \tag{2}$$

The equation for the capital market line is, in the form $y = \text{mx} + c$:

$$E(R_i) = E(R_f) + \frac{\sigma_i}{\sigma_m}\left(E(R_m) - E(R_f)\right) \tag{3}$$

The question now becomes: how can we use the CML to find the expected return on a bundle of risky assets?

The fundamental insight of the Capital Asset Pricing Model, developed by Sharpe (1964), Lintner (1965) and Treynor (1961), is once each investor holds proportional amounts of risky assets, the only thing they need to know is the covariance of their portfolio with the market portfolio. CAPM allows the investor to split risk into diversifiable risk and fundamental risk, with only the fundamental risk playing a part in the pricing of a stock.

The thinking goes like this. First, assume there are no transactions costs or informational deficiencies in the market. Everyone knows everything, and can transact costlessly. Investors therefore hold the same opinions about which assets are good buys, sells, or holds. Assume further that investors only want to hold stocks for a definite, and common, investment horizon.

Now imagine you hold the market portfolio. You add some risky asset, i, to the portfolio, so you have some share of your total wealth α invested in asset i, and the rest of your wealth, $(1-\alpha)$, invested in the market portfolio. For each different α (0.05, for example, is 5% of your wealth in asset i), you get different combinations of risk and return. Your expected return on a portfolio is

$$E(R_\alpha) = \alpha \mathrm{E}(R_i) + (1-\alpha)\, E(R_m), \tag{4}$$

Where $E(R_m)$ is the expected return on the market portfolio.

What is the risk, or standard deviation, of this portfolio?

$$\sigma_\alpha = \left(\alpha^2 \sigma^2{}_i + (1-\alpha)\,\sigma^2{}_m + 2\,\alpha(1-\alpha)\,\sigma_{\mathrm{im}}\right)^{1/2} \tag{5}$$

Which is just the square root of the square of the expected return, cleand up a little.

What is the expected return as we vary α? We must differentiate with respect to α to find this out. Essentially we are varying the value of α to find the optimal portfolio.

$$\frac{\mathrm{dE}(R_\alpha)}{\mathrm{d}\alpha} = E(R_i) - E(R_m). \tag{6}$$

Repeating this calculation with the standard deviation, we have

$$\frac{\mathrm{d}\sigma_\alpha}{\mathrm{d}\alpha} = \alpha\sigma_i{}^2 + (\alpha-1)\,\sigma_m{}^2 \tag{7}$$

Now go back to CAPM: what do we have? We see that at the market point in the figure above (point M), the value of α is zero: those investors are holding exactly the market portfolio. If we evaluate the derivative we just created at $\alpha = 0$, we get

$$\frac{\mathrm{d}\sigma_\alpha}{\mathrm{d}\alpha} = \frac{\alpha\sigma_i{}^2 + (\alpha-1)\,\sigma^2{}_m + (1-2\,\alpha)\,\sigma_{\mathrm{im}}}{\sigma_\alpha} \tag{8}$$

And, when $\alpha = 0$,

$$\frac{\mathrm{d}\sigma_\alpha}{\mathrm{d}\alpha}\Big|_{\alpha=0} = \frac{\sigma_i - \sigma_{\mathrm{im}}{}^2}{\sigma_\alpha}. \tag{9}$$

We know that

$$\frac{\mathrm{dE}(R_\alpha)}{\mathrm{d}\sigma_\alpha} = \frac{\mathrm{dE}(R_\alpha)/\mathrm{d}\alpha}{\mathrm{d}\sigma_\alpha/\mathrm{d}\alpha}, \tag{10}$$

so,

$$\frac{\mathrm{dE}(R_\alpha)}{\mathrm{d}\sigma_\alpha}\Big|_{\alpha=0} = \frac{(E(R_i) - E(R_m))\,\sigma_{\mathrm{im}}}{\sigma_{\mathrm{im}} - \sigma_m{}^2}. \tag{11}$$

The equation above is the slope of the market portfolio at the market point, M. You should see that this expression must equal the slope of the CML. Recall the slope of the CML is $\left(E\left(R_m - E\left(R_f\right)\right)\right/\sigma_m$. So

$$\frac{\left(E(R_m) - E\left(R_f\right)\right.}{\sigma_m} = \frac{(E(R_i) - E(R_m))\,\sigma_{\mathrm{im}}}{\sigma_{\mathrm{im}} - \sigma_m{}^2} \tag{12}$$

You can rearrange this to get

$$E(R_i) = E\left(R_f\right) + \frac{\sigma_{\mathrm{im}}}{\sigma_m{}^2}\left(E(R_m) - E\left(R_f\right)\right). \tag{13}$$

Now a little digression.

When you want to plot a line through a sequence of points, most economists will tell you to draw the line such that it minimises the sum of squard errors between the line and the points the line is supposed to describe. Say you have a line you want to fit, and it is of the form

$$y_t = \alpha_t + \beta \mathrm{x}_t + \epsilon_t, \tag{14}$$

When you estimate this line, you will receive a slope value for β, which we'll call $\hat{\beta}$, of the form

$$\hat{\beta} = \frac{\sigma_{\mathrm{xy}}}{\sigma \mathrm{x}^2}. \tag{15}$$

When you want to fit a line like the CAPM line we derived above, you'll estimate

$$E(R_i) = E\left(R_f\right) + \beta\left(E(R_m) - E\left(R_f\right)\right) + \epsilon_i, \tag{16}$$

and the slope coefficient you will recieve will be

$$\hat{\beta} = \frac{\sigma_{\mathrm{im}}}{\sigma^2{}_m}. \tag{17}$$

This is the β market analysts and finance people talk of when measuring the fundamental risk of a stock. We can reduce the message of CAPM further though. CAPM says the β will measure the excess return on an asset $(E(R_i) - E\left(R_f\right)$, relative to the excess net return on the market, $E(R_m) - E\left(R_f\right)$.

In this world, the excess return on an asset is just β times the market excess return.

Example

Find the required return on a risky asset such as a stock, given a risk free rate of return of 5%, a market return of 11%, and a β of 2.

$E(R_i) = E\left(R_f\right) + \beta\left(E(R_m) - E\left(R_f\right)\right)$
$= 5 + 2\,(11 - 5)$
$= 17.$

Exercise

Find the β of a risky asset with return 2%, with risk free rate of 4%, and a market portfolio return of 12%.

In Perold (2004), we are given a treatment of diversification, correlation, and risk, from a historical and theoretical point of view. Perold is required reading for this lecture.