EC4024 Lecture 8. Arbitrage Pricing Theory  
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The Arbitrage Pricing theory, or APT, was developed to shore up some of the deficiencies of CAPM we discussed in at the end of the last lecture. In particular, CAPM only works when we make assumptions about preferences which don't make much sense: consumers only care about mean and standard deviations in their wealth if their preferences are quadratic, as Markowitz and Sharpe specified them. Returns must also be normally (that is, Gaussian) distributed. Finally and most importantly, people hold different beliefs, and these beliefs lead them to hold different portfolios. It is therefore not quite clear what the market portfolio actually is. In practice we would use a large stock index like the S&P 500, but this is not ideal.

CAPM shows us a world in which \( \beta \) is king, however, when investors hold different portfolios, the value of \( \beta \) changes. When you measure the market portfolio differently (say, by taking a different broad index of stocks and shares), you get a different result for \( \beta \).

APT was developed to shore up these deficiencies. Ross (1976), who developed APT, dropped the assumptions on preferences and strict maximisation. He kept the idea that firms and stocks are looking for profit maximising opportunities, and the market was hard to beat. Rather than evolving an equilibrium condition for the market from consumer preferences as Sharpe did, Ross snapped the market equilibrium onto the investors, merely assuming that the search for arbitrage would keep investors at or near the CAPM-derived equilibrium.

The big idea of APT is to look at which combinations of assets one would hold to rule out any arbitrage. Arbitrage is possible when two assets with the same risk have different returns. You can short the low return asset, go long on the other using the proceeds of the sale of the first, and in theory, reap infinite rewards with no risk to yourself.

**Single Factor APT**

Begin with a single 'factor', \( F \), or driver, which generates the return \( (r_i) \) we see on every asset \( i \), such that

\[
E(R_i) = a_i + b_i F + \epsilon_i
\]

(1)

What fills the role of a factor?

The market rate of return, \( M \), we talked about in the last lecture, or the rate of economic growth, or inflation, or some other macroeconomic factor. The point is, the 'factor' is system wide, and there is only one.

As usual when modelling, we have to make some simplifying assumptions. Assume the following:

\[
\begin{align*}
E(\epsilon_i) &= 0, \\
\epsilon_j &= \text{correl}(\epsilon_i, \epsilon_j), \\
3\epsilon_i &= \text{correl}(\epsilon_i, F) = 0, \\
E(F) &= 0
\end{align*}
\]

(2)

What do these conditions mean? Let's take them one by one. First, \( E(\epsilon_i) = 0 \). This means that the long run mean of the errors falls to zero, so we are assuming the law of large numbers holds in this model. OK. Second, \( \text{correl}(\epsilon_i, \epsilon_j) = \text{correl}(\epsilon_i, F) = 0 \). This says the comovement of two assets, \( i \) and \( j \), are not related, so the returns on each stock (and the errors we get in measuring them) are independent of one another. Third, \( \text{correl}(\epsilon_i, F) = 0 \). The errors in measurement are not correlated with the factor. This is a bit of a stretch. Why? Can you think of an example where this might not hold? Well, we have to assume it to get the model up off the ground, so let's do that. Fourth, the mean of the factor is zero.

Let's say there is no residual risk, so \( \epsilon_i = 0 \).

Then returns would be calculated via

\[
E(R_i) = a_i + b_i E(F)
\]

(3)

Let's say we invest some fraction of our wealth \( \lambda \) in asset \( i \) and \( (1-\lambda) \), everything else, in asset \( j \). What is the return on this portfolio, \( P \)?

\[
E(R_P) = \lambda(a_i + b_i F) + (1-\lambda)(a_j + b_j F)
\]

\[
= \lambda(a_i - a_j) + \lambda b_j - b_i F.
\]

(4)

Let's try to weight the portfolio to make some more money out of it. Say the weight, \( \lambda^* \), looks like this:

\[
\lambda^* = \frac{b_j}{b_j - b_i}.
\]

(5)

Don't forget the portfolio has zero exposure to risk: the coefficient on \( F \) is zero, so we have

\[
E(R_P) = \frac{b_j}{b_j - b_i} (a_i - a_j) + a_j = E(R_F).
\]

(6)

Just rearrange the terms of the equation above, and we've got

\[
\frac{a_j - E(R_F)}{b_j} = \frac{a_i - E(R_F)}{b_i}
\]

(7)
Let's call this ratio \( \theta \).

\[
\theta = \frac{a_i - E(R_F)}{b_i}.
\]

(8)

We know \( E(R_i) = a_i \) from our discussion above, so plugging this into our formula we've just derived, we have the basic APT formula.

\[
E(R_i) = E(R_F) + b_i \theta.
\]

(9)

**Interpreting \( \theta \).**

Let's say you have a portfolio with \( b_p = 1 \), that is, it has one unit of 'factor risk' associated with it. Plug this into the formula above, and we get

\[
E(R_p) - E(R_F) = \theta.
\]

\( \theta \) is the excess return on a portfolio with a one unit of factor risk.

Now, we open ourselves up to a world of hurt (through matrix algebra) when we consider situations where \( \epsilon_i \neq 0 \). So we won't go there. Ross (1976) does, so read him if you want to know how he does it.

We can also consider multi-factor models, which I'll show you by way of an example.

**Example**

Suppose for some assets X, Y, and Z, the following two factors have these influences. You can think of the two factors as inflation and GDP growth, or something like that.

Suppose there are two factors:

1. Unanticipated market return, \( E(R_M) \)
2. Unanticipated inflation, \( E(R_I) \).

We have

\[
E(R_i) = E(R_M) + \beta_1 E(R_F) + \beta_2 E(R_I) + \epsilon_i.
\]

(10)

Say \( E(R_2) = 0.05 \), \( E(R_M) - E(R_F) = 0.08 \), \( E(R_F) - E(R_I) = -0.02 \).

What are the returns on the factor portfolios?

\[
\begin{align*}
E(R_M) &= (0.05 + 0.08) + E(R_F) \\
E(R_I) &= (0.05 - 0.02) + E(R_F)
\end{align*}
\]

(11)

First, let's assume there is only factor risk associated with an asset \( q \) such that

\[
E(R_q) = E(R_q) + F_1 + F_2.
\]

(12)

APT requires that it's expected rate of return will be

\[
E(R_q) = E(R_q) + E(R_F) + E(R_I)
\]

(13)

\[ = 0.05 + (1.0)(0.08) + (1.0)(-0.02) = 11 \%
\]

If the risk free rate was given by, say, 9%, then there is an arbitrage opportunity.

**Exercise**

I give you 100 euros and let you know about the situation as described in the example above. How much would you invest?

**How to use APT**

To use APT, you'll need to follow these steps:

1. Identify the factors
2. Estimate the factor weights on each asset
3. Estimate the factor premia.

We'll take these in turn.

First, we have to find the factors. Candidates are things like

1. changes in gdp growth,
2. changes in the T-bill yield as a proxy for inflation,
changes in the yield spread between bills and bonds of interest,
4. changes in the default premium of some corporate bonds,
5. changes in oil prices, again as a proxy for inflation
and so on

Use factor analysis:
1. estimate the covariance of asset returns, and on the basis of this, extract the 'factors' from the covariance matrix.

Use data mining:
explore, using a computer, different mixtures of portfolios, to find those whose returns can be used as factors.

2. Factor Weights.

Given the factors, regress past asset prices on the factors to estimate factor weights

3. Factor Premia.

You know the factors to look at, and you've figured out the weights of individual assets. Now it's a simple matter to construct a factor portfolio. Just plug one into the other using the formula above.

4. A PT Pricing.

APT gives the return on asset i as

\[ \text{E}(R_i) = \text{E}(R_q) + b_1(\text{E}(R_{F1}) - R_q) + ... + b_K(\text{E}(R_{FK}) - R_q). \]  

Summary: Strength and Weaknesses of APT

1. The model gives a reasonable description of return and risk.
2. Factors seem plausible.
3. No need to measure market portfolio correctly.
4. Model itself does not say what the right factors are.
5. Factors can change over time.

Differences between APT and CAPM's

APT is based on the factor model of returns and the "approximate arbitrage" argument.

CAPM's are based on investors' portfolio demand and equilibrium arguments.

Differences between APT and Arbitrage - Free Pricing

APT uses "approximate arbitrage" to approximately price (almost) "all" assets.

Arbitrage-free pricing (e.g. option pricing) uses strict arbitrage to price assets that can be replicated exactly.

Further Reading
